3D Composite Transformation

It is the combination of more than one transformation.

Contents

Transformation in 3D

- Transformation Matrix

$$\begin{bmatrix} A & D & G & J \\ B & E & H & K \\ C & F & I & L \\ 0 & 0 & 0 & S \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} 3\times 3 & 3\times 1 \\ \hline 1\times 3 & 1\times 1 \end{array}\right]$$

3×3 : Scaling, Reflection, Shearing, Rotation

3×1 : Translation

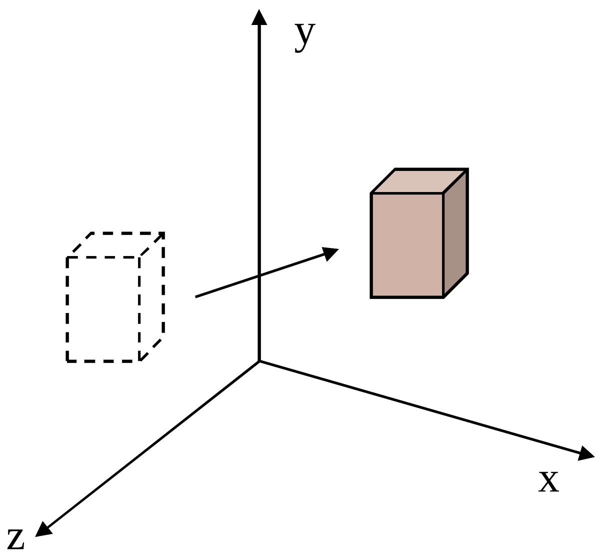
1×1 : Uniform global Scaling

1×3 : Homogeneous representation

3D Translation

- Translation of a Point

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$

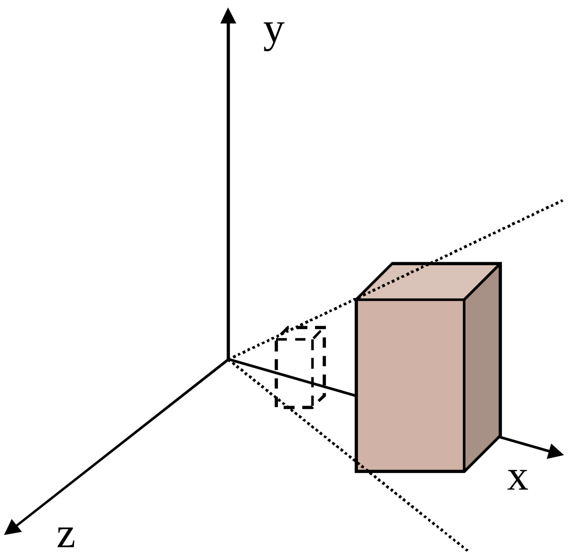


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Scaling

- Uniform Scaling

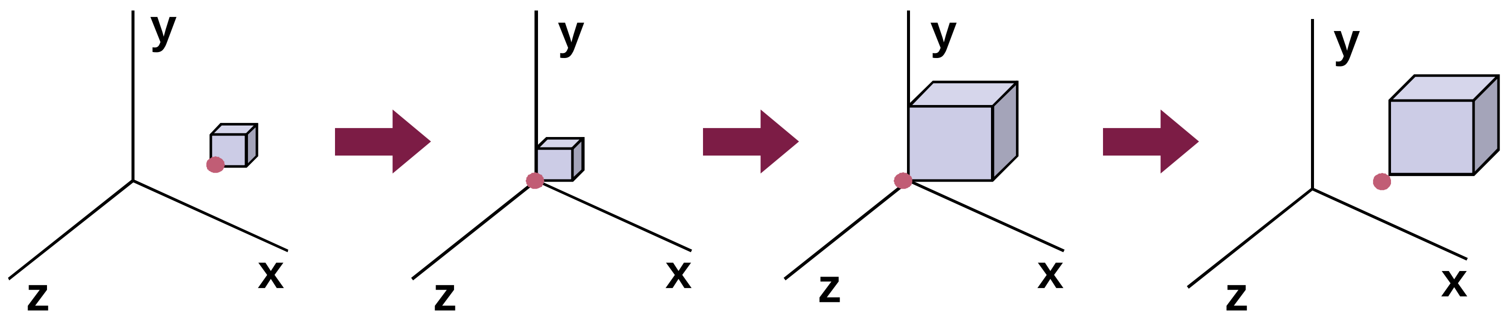
$$x' = x \cdot s_x\,, \quad y' = y \cdot s_y\,, \quad z' = z \cdot s_z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Relative Scaling

- Scaling with a Selected Fixed Position



Original position | Translate | Scaling | Inverse Translate

$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Rotation

- Coordinate-Axes Rotations
  - X-axis rotation
  - Y-axis rotation
  - Z-axis rotation

- General 3D Rotations
  - Rotation about an axis that is parallel to one of the coordinate axes
  - Rotation about an arbitrary axis

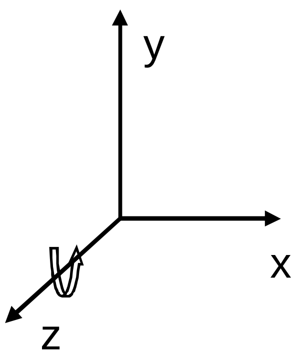
Coordinate-Axes Rotations

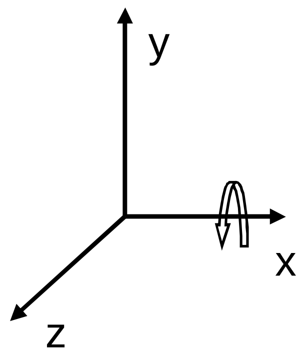
- Z-Axis Rotation
- X-Axis Rotation
- Y-Axis Rotation

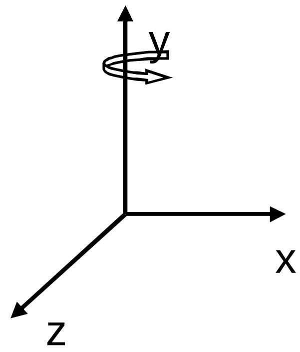
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

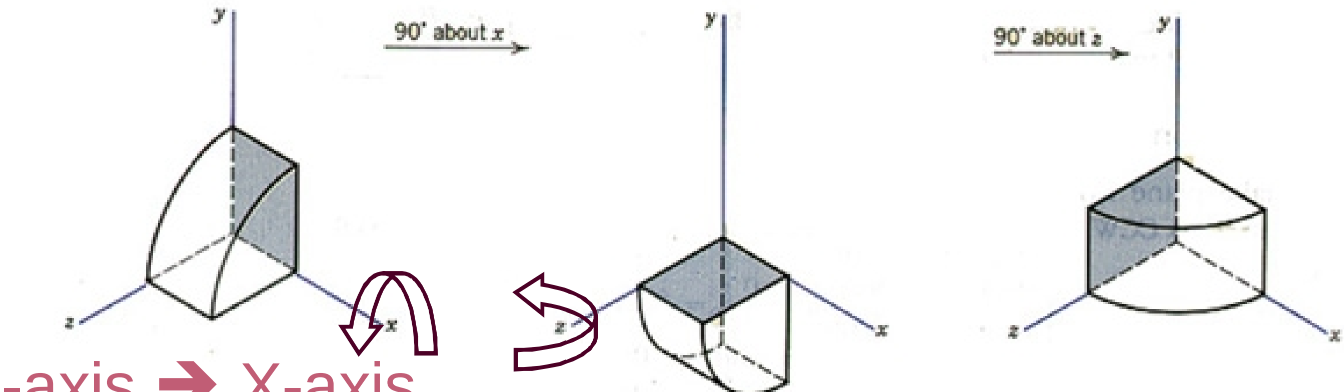




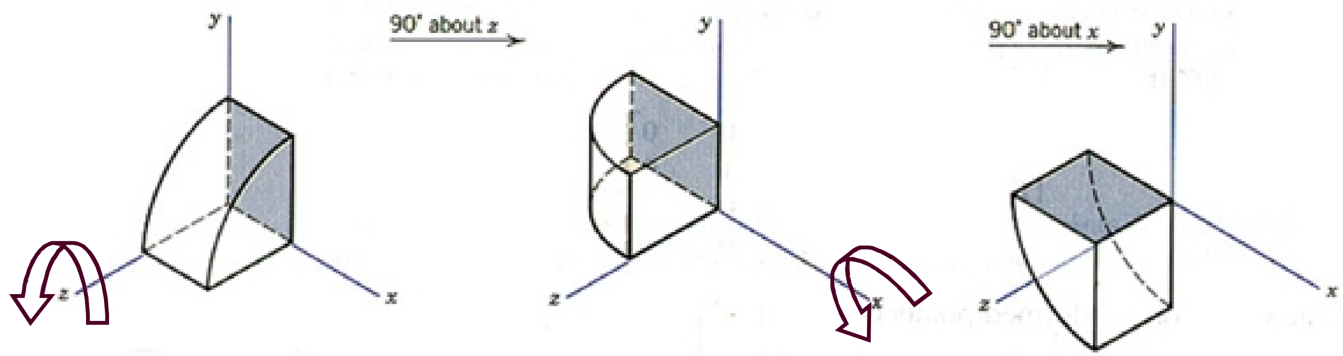


Order of Rotations

- Order of Rotation Affects Final Position
  - X-axis ➔ Z-axis

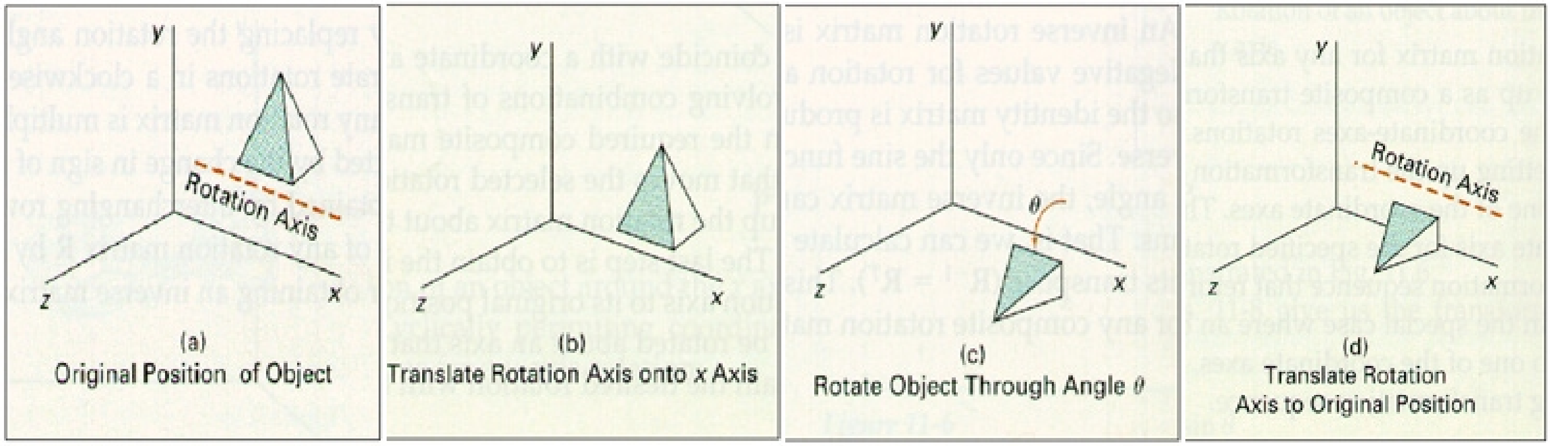


  - Z-axis ➔ X-axis



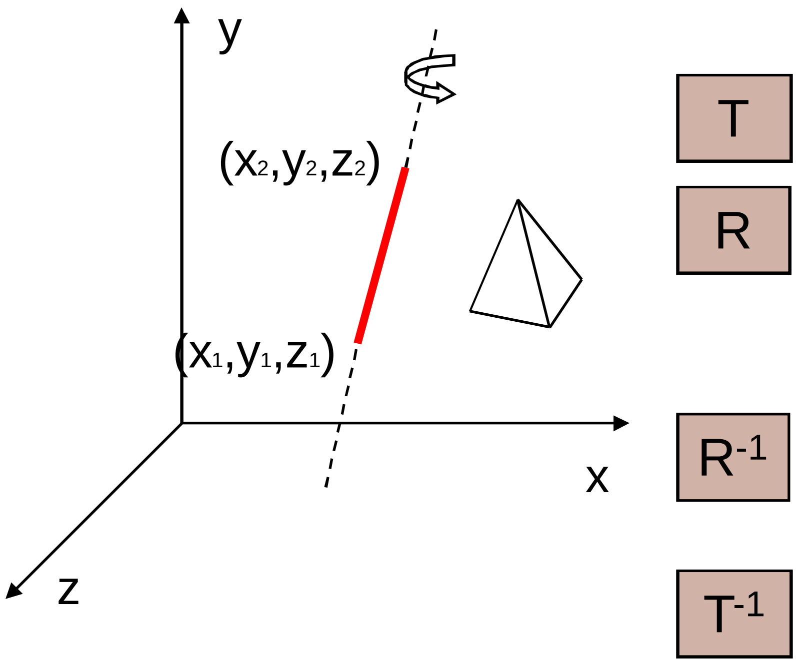
General 3D Rotations

- Rotation about an Axis that is Parallel to One of the Coordinate Axes
  - **Translate** the object so that the rotation axis <u>coincides with the parallel coordinate axis</u>
  - Perform the specified **rotation** about that axis
  - **Translate** the object so that the rotation axis is <u>moved back to its original position</u>



General 3D Rotations

- Rotation about an Arbitrary Axis



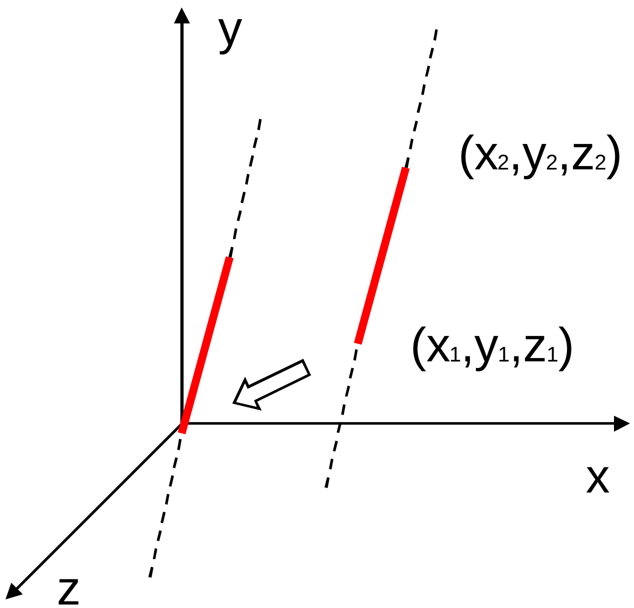
Basic Idea

1. **Translate (x1, y1, z1) to the origin**
2. **Rotate (x'2, y'2, z'2) on to the z axis**
3. **Rotate the object around the z-axis**
4. **Rotate the axis to the original orientation**
5. **Translate the rotation axis to the original position**

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

General 3D Rotations

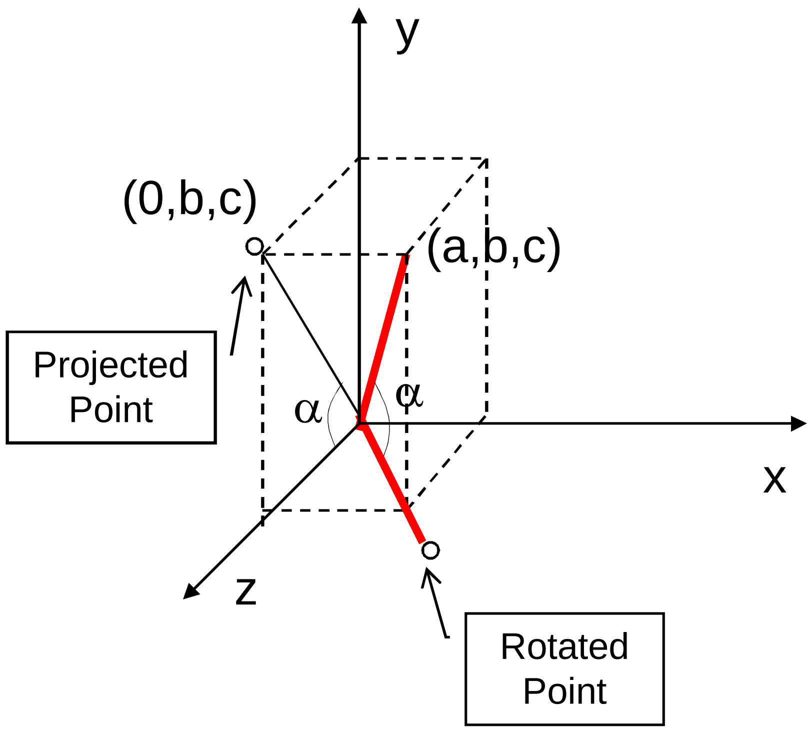
- Step 1. Translation



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General 3D Rotations

- Step 2. Establish $[T_R]^{\alpha}_{x}$ *x* axis



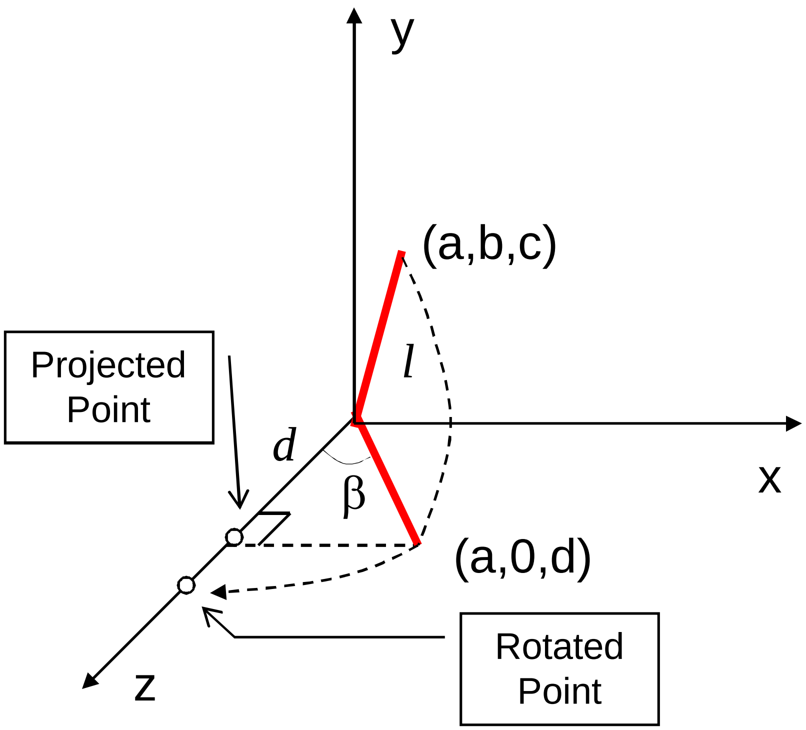
$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Axis Rotation

- Step 3. Rotate about *y* axis by ϕ



$$\sin\beta = \frac{a}{l}, \qquad \cos\beta = \frac{d}{l}$$

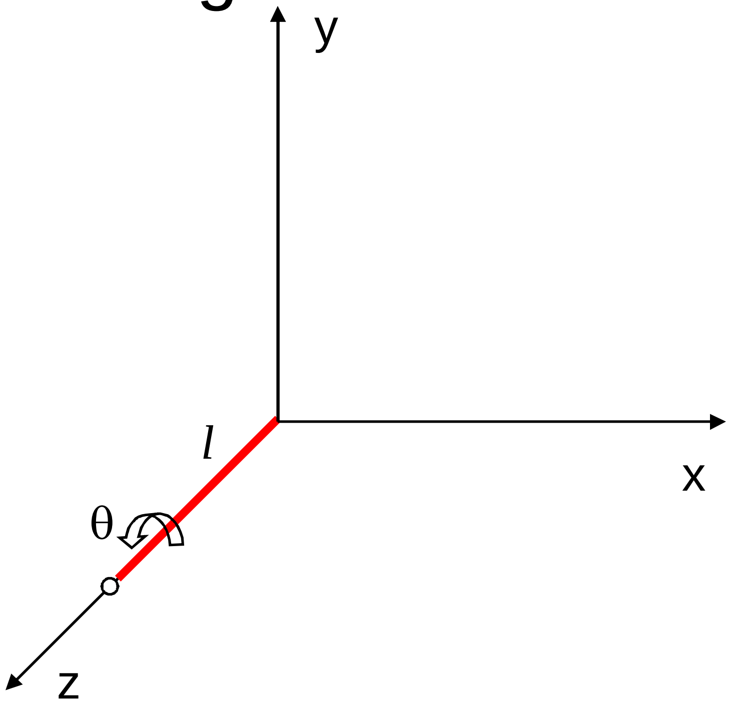
$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Axis Rotation

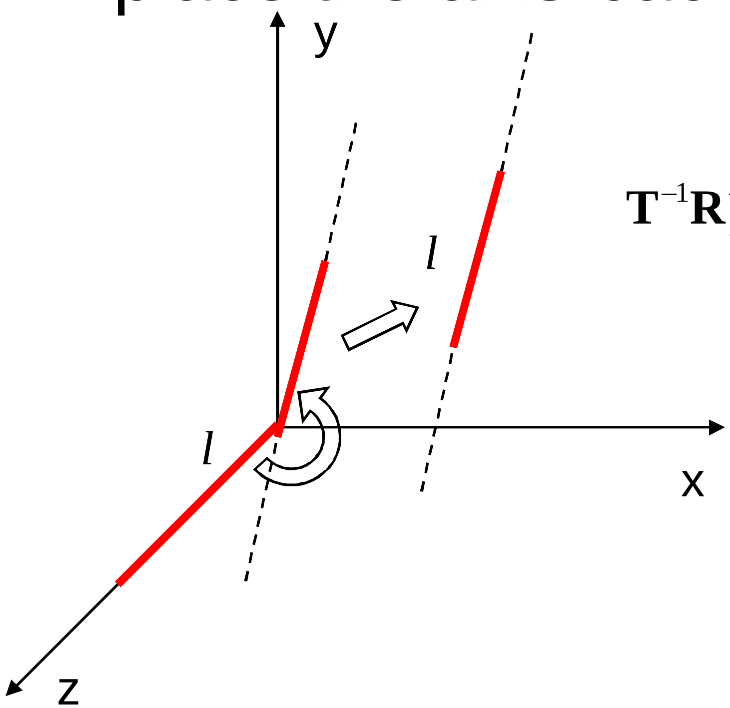
- Step 4. Rotate about *z* axis by the desired angle θ



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Axis Rotation

- Step 5. Apply the reverse transformation to place the axis back in its initial position

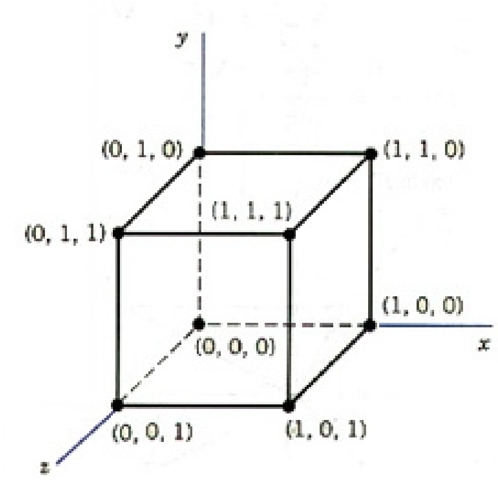


$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)=\begin{bmatrix}1&0&0&-x_1\\0&1&0&-y_1\\0&0&1&-z_1\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\cos\beta&0&\sin\beta&0\\0&1&0&0\\-\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

Example

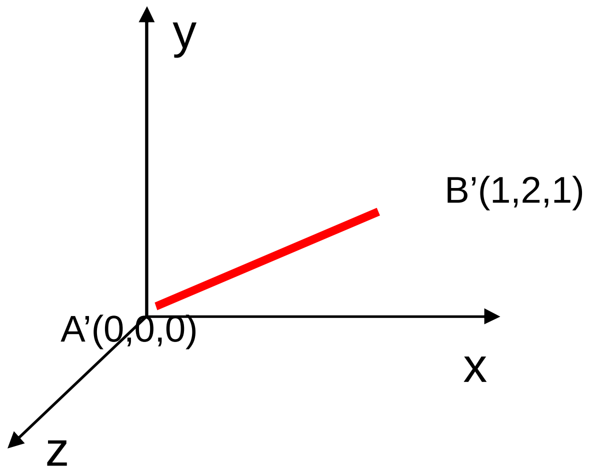
Find the new coordinates of a unit cube 90º-rotated about an axis defined by its endpoints A(2,1,0) and B(3,3,1).



A Unit Cube

Example

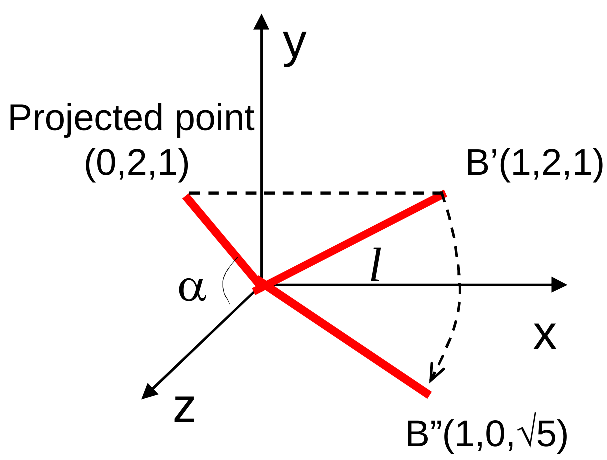
- Step1. Translate point A to the origin



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 2. Rotate axis *A'B'* about the x axis by and angle α, until it lies on the *xz* plane.



$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

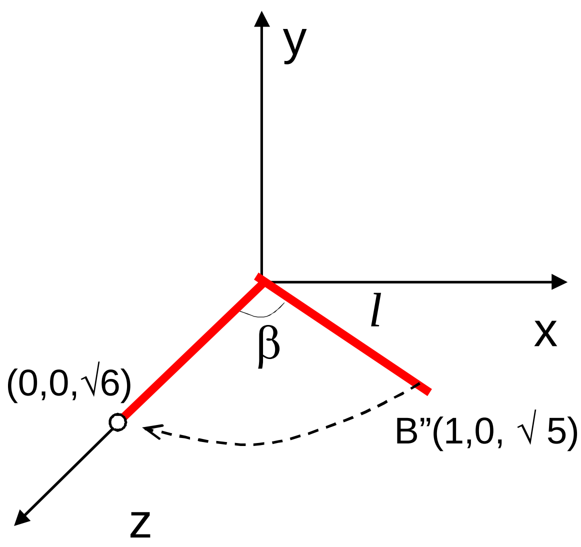
$$\cos\alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$l = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 3. Rotate axis *A'B''* about the y axis by and angle ϕ, until it coincides with the z axis.



$$\sin\beta = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$$\cos\beta = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 4. Rotate the cube 90° about the z axis

$$\mathbf{R}_z(90°) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Finally, the concatenated rotation matrix about the arbitrary axis AB becomes,

$$\mathbf{R}(\theta) = \mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(90°)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

Example

$$\mathbf{R}(\theta)=\begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{\sqrt{5}}{5}&\frac{2\sqrt{5}}{5}&0\\0&-\frac{2\sqrt{5}}{5}&\frac{\sqrt{5}}{5}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{30}}{6}&0&\frac{\sqrt{6}}{6}&0\\0&1&0&0\\-\frac{\sqrt{6}}{6}&0&\frac{\sqrt{30}}{6}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}\frac{\sqrt{30}}{6}&0&-\frac{\sqrt{6}}{6}&0\\0&1&0&0\\\frac{\sqrt{6}}{6}&0&\frac{\sqrt{30}}{6}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{\sqrt{5}}{5}&-\frac{2\sqrt{5}}{5}&0\\0&\frac{2\sqrt{5}}{5}&\frac{\sqrt{5}}{5}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0.166&-0.075&0.983&1.742\\0.742&0.667&0.075&-1.151\\-0.650&0.741&0.167&0.560\\0&0&0&1\end{bmatrix}$$

Example

- Multiplying $\mathbf{R}(\theta)$ by the point matrix of the original cube

$$[\mathbf{P}'] = \mathbf{R}(\theta) \cdot [\mathbf{P}]$$

$$[\mathbf{P}'] = \begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.650 & 1.667 & 1.834 & 2.816 & 2.725 & 1.742 & 1.909 & 2.891 \\ -0.558 & -0.484 & 0.258 & 0.184 & -1.225 & -1.151 & -0.409 & -0.483 \\ 1.467 & 1.301 & 0.650 & 0.817 & 0.726 & 0.560 & -0.091 & 0.076 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Rotations with Quaternions

- Quaternion
  - Scalar part s + vector part $\mathbf{v} = (a, b, c)$
  - Real part + complex part (imaginary numbers i, j, k)

$$q = (s, \ \mathbf{v}) = s + ai + bj + ck$$

- Rotation about any axis
  - Set up a unit quaternion ($\mathbf{u}$: unit vector)

$$s = \cos\frac{\theta}{2}, \ \ \mathbf{v} = \mathbf{u}\sin\frac{\theta}{2}$$

  - Represent any point position $\mathbf{P}$ in quaternion notation ($\mathbf{p} = (x, y, z)$)

$$\mathbf{P} = (0, \ \mathbf{p})$$

Rotations with Quaternions

- Carry out with the quaternion operation ($q^{-1}=(s, -\mathbf{v})$)

$$\mathbf{P}' = q\mathbf{P}q^{-1}$$

- Produce the new quaternion

$$\mathbf{P}' = (0,\ \mathbf{p}')$$

$$\mathbf{p}' = s^2\mathbf{p} + \mathbf{v}(\mathbf{p}\cdot\mathbf{v}) + 2s(\mathbf{v}\times\mathbf{p}) + \mathbf{v}\times(\mathbf{v}\times\mathbf{p})$$

- Obtain the rotation matrix by quaternion multiplication

$$\mathbf{M}_R(\theta) = \mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)$$

$$= \begin{bmatrix} 1-2b^2-2c^2 & 2ab-2sc & 2ac+2sb \\ 2ab+2sc & 1-2a^2-2c^2 & 2bc-2sa \\ 2ac-2sb & 2bc+2sa & 1-2a^2-2b^2 \end{bmatrix}$$

- Include the translations: $\mathbf{R}(\theta) = \mathbf{T}^{-1}\mathbf{M}_R(\theta)\mathbf{T}$

Example

- Rotation about z axis
  - Set the unit quaternion: $s = \cos\frac{\theta}{2}, \; \mathbf{v} = (0,\ 0,\ 1)\sin\frac{\theta}{2}$
  - Substitute $a=b=0$, $c=\sin(\theta/2)$ into the matrix:

$$\mathbf{M}_R(\theta) = \begin{bmatrix} 1-2\sin^2\frac{\theta}{2} & -2\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 0 \\ 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 1-2\sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

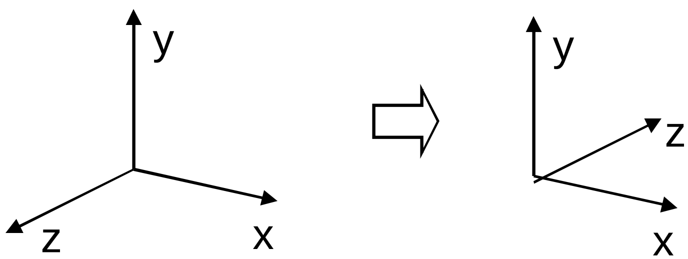
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1-2\sin^2\frac{\theta}{2} = \cos\theta$$

$$2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta$$

Other Transformations

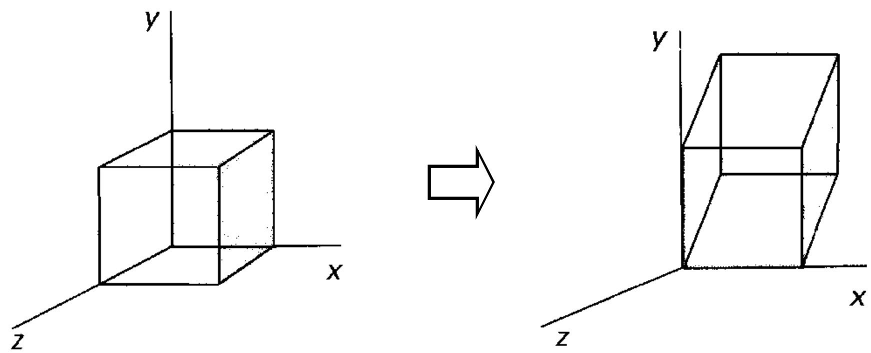
- Reflection Relative to the xy Plane



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \text{[redacted]} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

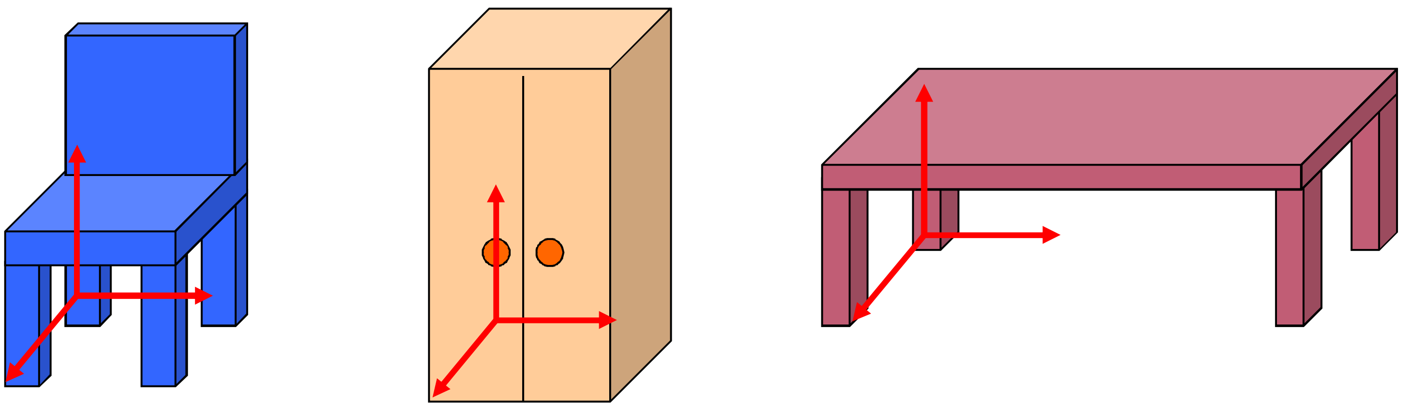
- Z-axis Shear

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \text{[redacted]} & 0 \\ 0 & 1 & \text{[redacted]} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Coordinate Transformations

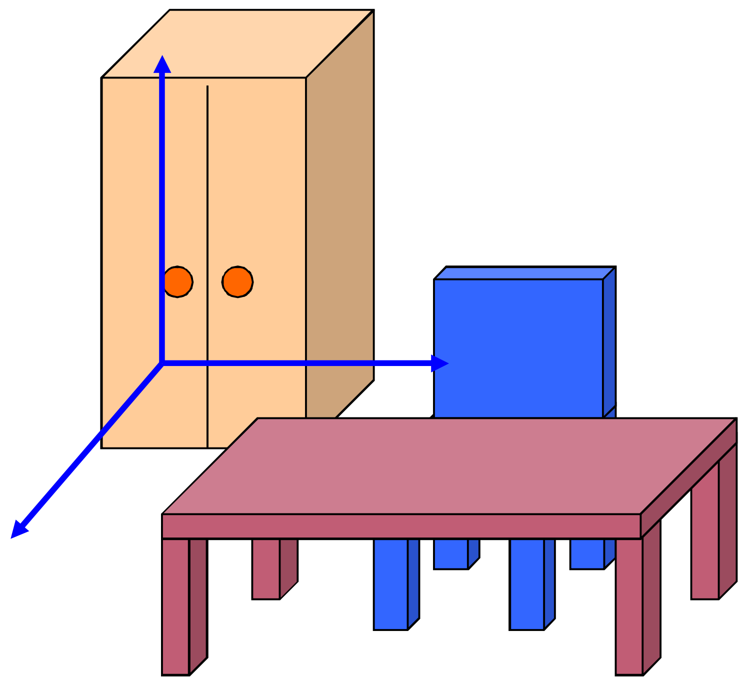
- Multiple Coordinate System
  - Local (modeling) coordinate system
  - World coordinate scene



Local Coordinate System

Coordinate Transformations

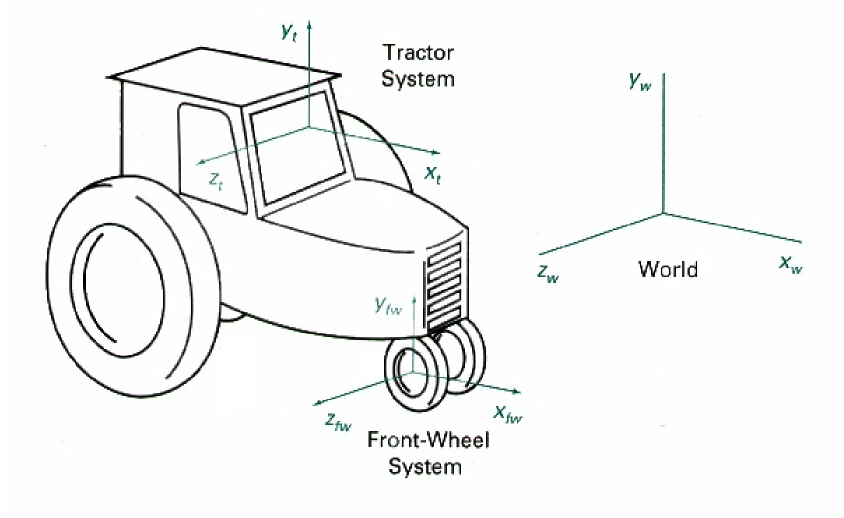
- Multiple Coordinate System
  - Local (modeling) coordinate system
  - World coordinate scene



Word Coordinate System

Coordinate Transformations

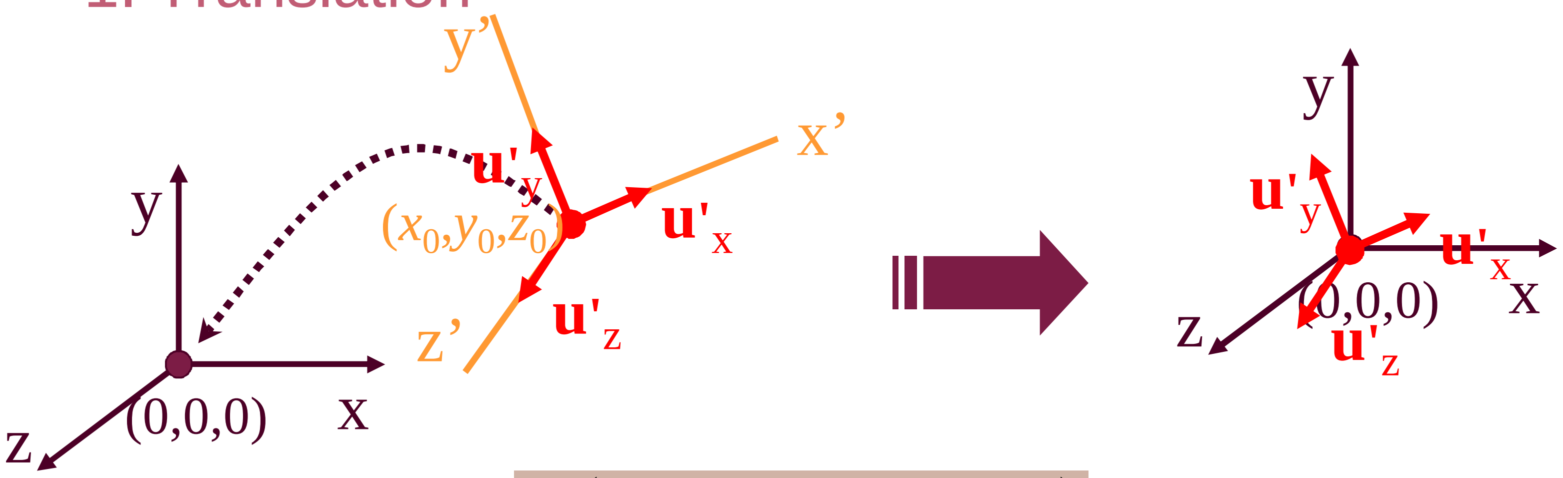
- Example – Simulation of Tractor movement
  - As tractor moves, **tractor coordinate system** and **front-wheel coordinate system** <u>move</u> in world coordinate system
  - **Front wheels** <u>rotate</u> in wheel coordinate system



Coordinate Transformations

- Transformation of an Object Description from One Coordinate System to Another
- Transformation Matrix
  - Bring the two coordinates systems into alignment

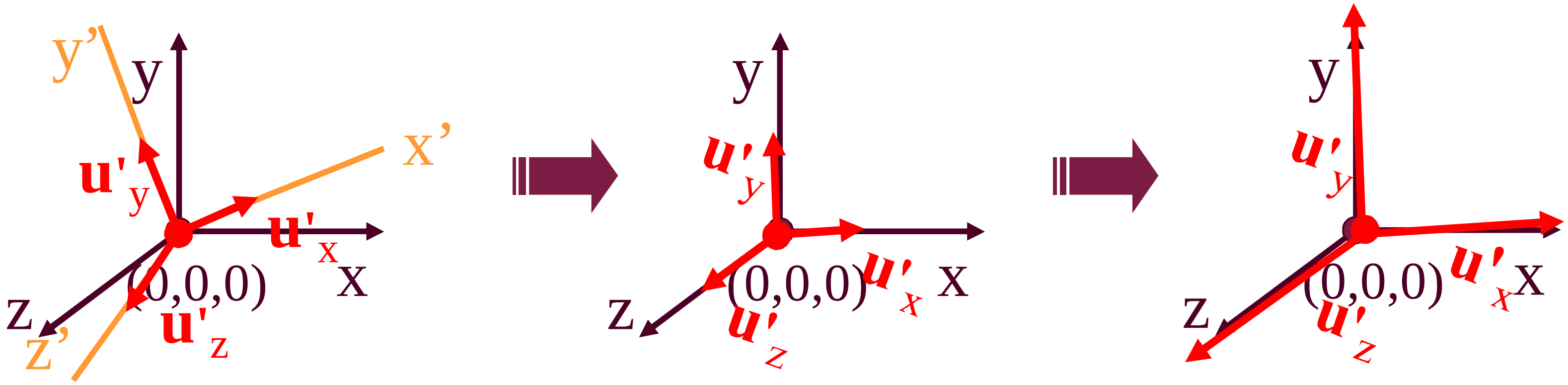
1. Translation



$$\mathbf{T}(-x_0, -y_0, -z_0)$$

Coordinate Transformations

2. Rotation & Scaling



$$
\mathbf{R} = \begin{bmatrix} u'_{x_1} & u'_{x_2} & u'_{x_3} & 0 \\ u'_{y_1} & u'_{y_2} & u'_{y_3} & 0 \\ u'_{z_1} & u'_{z_2} & u'_{z_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$